Notes and Hints and Solutions for HW due 09-03-09

1) a) $\lim_{x\to 8} 7 = 7$ b) $\lim_{x\to 0+} \pi = \pi$ Both a and b are *constant functions*.

c) $\lim_{x\to -2} 3x = -6$ d) $\lim_{y\to 3+} 12y = 36$ Pretty straightforward.

2) a) $\lim_{x\to 5} x/|x| = 5/5 = 1$ b) $\lim_{x\to -5} x/|x| = -5/5 = -1$

c) $\lim_{x\to 0^+} x/|x| = 0^+/0^+ = +1$ d) $\lim_{x\to 0^-} x/|x| = 0^-/0^+ = -1$

Recall that 0^+ and 0^- is notation I just created. They mean really small (close to 0) numbers that aren't themselves 0. When working with such numbers it is important to keep track of the **signs** of the little guys ….

3) Given that $\lim_{x\to a} f(x) = 2$; $\lim_{x\to a} g(x) = -4$; and $\lim_{x\to a} h(x) = 0$ …

a) $\lim_{x\to a}\left[f(x)+2g(x)\right] = \lim_{x\to a} f(x) + 2\lim_{x\to a} g(x) = 2 + (2)(-4) = -6$

b) $\lim_{x\to a}\left[h(x)-3g(x)+1\right] = \lim_{x\to a} h(x) - 3\lim_{x\to a} g(x) + \lim_{x\to a}(1) = 0-(3)(-4)+1 = 13$

c) $\lim_{x\to a}\left[f(x)g(x)\right] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = (2)(-4) = -8$

d) $\lim_{x\to a}\left[g(x)\right]^2 = \left[\lim_{x\to a} g(x)\right]^2 = (-4)^2 = 16$

e) $\lim_{x\to a}\sqrt[3]{6+f(x)} = \sqrt[3]{\lim_{x\to a}\left(6+f(x)\right)} = \sqrt[3]{6+2} = 2$

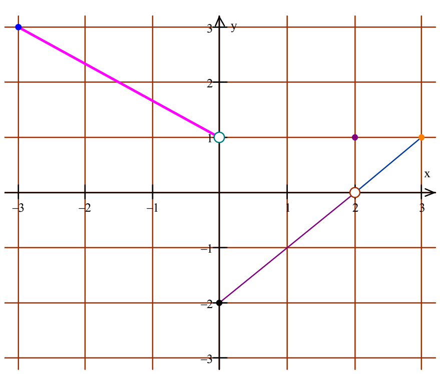
f) $\lim_{x\to a}\left[2/g(x)\right] = \left[\lim_{x\to a} 2\right] \Big/ \left[\lim_{x\to a} g(x)\right] = 2/(-4) = -1/2$

g) $\lim_{x\to a}\dfrac{3f(x)-8g(x)}{h(x)} = \dfrac{\lim_{x\to a}\left[3f(x)-8g(x)\right]}{\lim_{x\to a} h(x)}$ and *h(x)* approaches 0 while the numerator does not. So the limit will not exist.

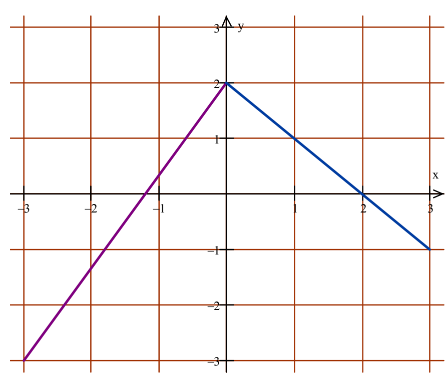
h) Same reason as (g) – the limit will not exist.

4)

f(x):



g(x):



a) $\lim_{x\to 2}\left[f(x)+g(x)\right] = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x) = 0+0 = 0$

b) $\lim_{x\to 0}\left[f(x)+g(x)\right] = \lim_{x\to 0} f(x) + \lim_{x\to 0} g(x) = dne$ **because** f(x) does not have a limit as $x \to 0$

c) $\lim_{x\to 0+}\left[f(x)+g(x)\right] = \lim_{x\to 0+} f(x) + \lim_{x\to 0+} g(x) = (-2)+2 = 0$

d) $\lim_{x\to 0-}\left[f(x)+g(x)\right] = \lim_{x\to 0-} f(x) + \lim_{x\to 0-} g(x) = 1+2 = 3$

e) $\lim_{x\to 2}\left[\frac{f(x)}{1+g(x)}\right]=\frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2}\left[1+g(x)\right]}=\frac{0}{1+2}=0$

f) $\lim_{x\to 2}\left[\frac{1+g(x)}{f(x)}\right]=\frac{\lim_{x\to 2}\left[1+g(x)\right]}{\lim_{x\to 2} f(x)}=\frac{3}{dne}=dne$

g) $\lim_{x\to 0+}\left[\sqrt{f(x)}\right]=\sqrt{\lim_{x\to 0+} f(x)}="\sqrt{-2}"=dne$

h) $\lim_{x\to 0-}\left[\sqrt{f(x)}\right]=\sqrt{\lim_{x\to 0-} f(x)}=\sqrt{1}=1$

So – you are just applying those limit laws to break the complicated expressions down into simpler expressions that we know (or can evaluate easily).

There are several things that can result from this:
* You get a number (=you are done)
* You get something that is approaching $0/0$ or ∞/∞. YOU DON'T KNOW ANYTHING IN THIS CASE. If you are dealing with polynomials, cancel common factors. There are also some specialized limits we will learn later.
* You get something approaching 0 only in the denominator. THEN – you keep track of signs so you can decide whether to write $= dne$, $=\infty$, or $=-\infty$
* If any of the pieces has a limit of '*dne*', your answer is probably *dne* unless there is some weird canceling-out thing going on.

5) $\lim_{x\to 2-}\frac{(y-1)(y-2)}{y+1}="substitute"=\frac{(1)(0)}{3}=0$

6) $\lim_{x\to 3}\frac{x^2-2x}{x+1}="substitute"=\frac{9-6}{4}=3/4$

7) $\lim_{x\to 4}\frac{x^2-16}{x-4}="cancelfactors"=\lim_{x\to 4}\frac{(x-4)(x+4)}{(x-4)}=\lim_{x\to 4}\frac{x+4}{1}=8$. How do we know to cancel common factors? Because we **first** tried substituting and we would get 0/0 …. YUCK!

8) $\lim_{x\to 0}\frac{6x-9}{x^3-12x+3}="substitution"=\frac{-9}{3}=-3$. The AP exam is timed, and you might not finish all of the M/C. So learn to work precisely, and then learn to work quickly. Here you would have wasted much time if you blindly factored before substituting …

9) $\lim_{x\to 1+}\frac{x^4-1}{x-1}=\lim_{x\to 1+} x^3+x^2+x+1="substitution"=4$… (surprised?). Once you know that (x-1) is a factor of the numerator, you can use long division or synthetic division to find the other factor. Personally – I don't use synthetic division, because I'm never certain I've remembered it correctly (don't have MUCH occasion to do this …) and because I am almost as fast (and absolutely certain) with long division.
NOTE: WolframAlpha can be used to check your work. I entered "factor x^4-1" and got the answer. Useful for **CHECKING YOUR STRUGGLE …**

10) $\lim_{t\to -2}\frac{t^3-8}{t-2}=\lim_{t\to -2}\left(t^2+t+1\right)=3$ (sigh) REAL 10) $\lim_{t\to -2}\frac{t^3+8}{t+2}=\lim_{t\to -2}\left(t^2-2t+4\right)=12$

11) $\lim_{x\to -1}\frac{x^2+6x+5}{x^2-3x-4}=\lim_{x\to -1}\frac{(x+5)}{(x-4)}="subs"=\frac{4}{-5}$

12) $\lim_{x\to 2}\frac{x^2-4x+4}{x^2+x-6}=\lim_{x\to 2}\frac{(x-2)}{(x+3)}=\frac{0}{5}=0$

13) $\lim_{t\to 2}\frac{t^3+3t^2-12t+4}{t^3-4t}=\lim_{t\to 2}\frac{t^2+5t-2}{t^2+2t}=\frac{12}{8}=\frac{3}{2}$. Note: you can easily factor denominator …

14) $\lim_{t\to 1}\frac{t^3+t^2-5t+3}{t^3-3t+2}=\lim_{t\to 1}\frac{t^2+2t-3}{t^2+t-2}=\lim_{t\to 1}\frac{t+3}{t+2}=\frac{4}{3}$

15) $\lim_{x\to 3^+}\frac{x}{x-3}="bot-to-0"=+\infty$

16) $\lim_{x\to 3^-}\frac{x}{x-3}=-\infty$ (15) and (16) are two sides of the same problem. When you encounter situations where the denominator goes to zero, and the numerator does not, it is important to consider the signs of the one-sided limits to understand the limit. Remember – if your denominator goes to zero, your answers are going to be (most probably) +infinity, -infinity, or DNE.

A note about the factoring limits … You will NOT be expected to factor cubics or above on the AP exam (with the possible exception of sum/difference of two cubes). You WILL be expected to be able to factor quadratics. If you need practice before May let me know!!

2.1 – No question about it, #17 and #18 are much more difficult than 15-16. I had assumed (oops) that there was a more gradual increase in difficulty. But let's see how we can think about these to try and get an answer ….

17) Sketch a possible graph of a f^n that has the following properties:

i) *f(x)* = 0 when *x* is an integer, and $f(x)\neq 0$ when *x* is NOT an integer.

ii) $\lim_{x\to\infty}f(x)=0$ and $\lim_{x\to-\infty}f(x)=0$

Clue (i) tells you this is going to be a weird one … right? I showed you a 'pathological' function a couple of days ago – and I am SURE that was the first time you saw anything like it in your life. So – here's #2 – and your comfort level is still going to be very low.

Let's look at clue (ii) then. Clue (ii) tells us that our desired function needs to have a horizontal asymptote in both the +infinity and –negativity directions. So – we'll have to choose the points in our function to fit a horizontal asymptote idea.

Now – we are told what the values of the function are for x = n where n is an integer. When we consider *f(x)* for *x* not an integer, we can choose any values, so let's try and match a function we KNOW to have a horizontal asymptote of $y = 0$. The function $y=1/x$ comes to mind. So all we need do is choose values of $1/x$ when *x* is not an integer. So -- $f(x)=\begin{cases}0, & \text{when } x \text{ is an integer}\\ 1/x, & \text{when } x \text{ is not an integer}\end{cases}$.

18) Sketch a possible graph of a f^n that has the following properties:

i) *f(x)* = 1 if *x* is a positive integer and $f(x)\neq 1$ when $x>0$ is NOT a positive integer.

ii) $f(x)=-1$ if *x* is a negative integer and $f(x)\neq -1$ if $x<0$ is not a negative integer

iii) $\lim_{x\to\infty}f(x)=1$ and $\lim_{x\to-\infty}f(x)=-1$

After reading the explanation above, try thinking about this one yourself for a little (10 minutes).

Matching our thinking above, obviously this might be another weird function, AND it has to have a horizontal asymptote of $y = 1$. Well … we could use any rational function where the degree of top and bottom polynomials is the same. But we can do a little better, because we saw that odd x over the absolute of x thing. But when x is not an integer we have to 'make it off center' a little – we'll add something to it that has a horizontal asymptote of 0. So let's try: $g(x) = \begin{cases} |x|/x, & x \text{ an integer } x \neq 0 \\ |x|/x + 1/x, & x \text{ not an integer} \end{cases}$. And of course this is NOT the only one that will work …

31) Suppose that $f(x)$ denotes a function such that $\lim_{t\to 0} f(1/t) = L$. What can be said about $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$?

This is our first exposure to a 'change of variable' strategy that is quite important in ♥calculus♥. Knowing what a limit does in one place doesn't tell me much about limits close by … and absolutely NOTHING about limits that are far away. So - is there some way that these two pieces of information are related?

If there is, one of the things we can try is to create a new variable x that is equal to our $1/t$. So …

Let $x = 1/t$.

But these are limit statements … So when t is approaching 0, what is happening to x? ? In other words, What is the limit of $1/t$ as t approaches 0? Well, it's +infinity OR –infinity depending on which direction we are approaching 0 from!!! So we have an idea!! So x approaching infinity means t approaches 0 and …

Let $x = 1/t$.

Then $\lim_{x\to\infty} f(x) = \lim_{\frac{1}{t}\to\infty} f(\frac{1}{t}) = \lim_{t\to 0+} f(\frac{1}{t}) = L$. And similarly for the other one …